Int. J. Solids Structures, 1968, Vol. 4, pp. 897 to 905. Pergamon Press. Printed in Great Britain

STABILITY OF MOTION OF IMPACT TOOLS

C. C. FU and B. PAUL

Ingersoll-Rand Research, Inc., Box 301, Princeton, New Jersey

Abstract—A method is developed for analyzing the rigid body dynamics of machines wherein a hammer is made to reciprocate within a housing and to periodically impact against a bit, moil, anvil or other energy absorbing member. The physical system is idealized and represented by a "floating" two-body model. It is found that two simple steady-state motions could exist. It was also found that the floating two-body model theoretically admits one stable steady-state solution which has the same period as the exciting force and exhibits only one impact per cycle of the steady-state motion. Stability regions of the simple steady-state solutions are determined.

A separate computer solution was constructed to predict the detailed history of the system's motion. The computer results indicate that the analytical steady-state solutions are stable only for extremely small disturbances.

1. INTRODUCTION

AN idealization of a reciprocating hammer impact tool is shown schematically in Fig. 1, where m_1 represents the tool case, m_2 represents the hammer, H is a constant down-force, the actuator provides a steady force P_0 and an oscillatory force of amplitude P_1 and frequency $\omega/2\pi$, acting between the case and hammer. The hammer would, in a real tool, impact against a bit. The bit has been replaced, in the idealization, by an energy sink, the details of which will be described later.

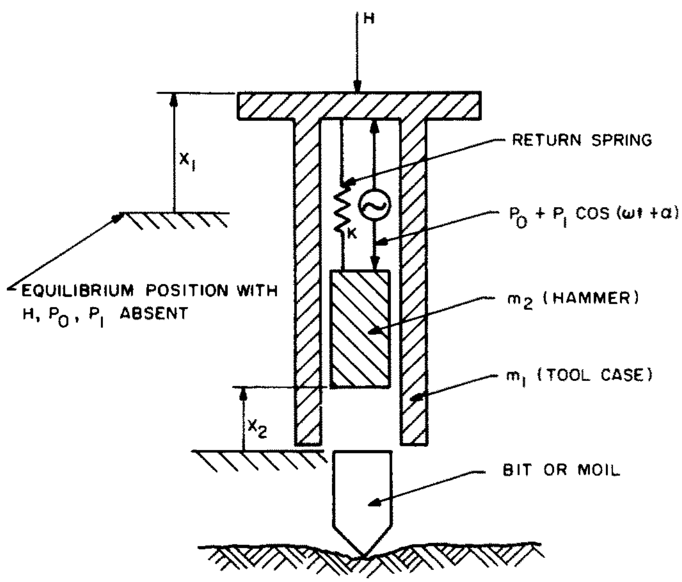

FIG. 1. Schematic of reciprocating hammer impact tool.

Because the impact occurs when the absolute displacement of the hammer reaches a specified value, rather than at a predetermined time, the problem is essentially nonlinear.

However, it is possible to construct synchronous (that is, of the same period as the exciting force) steady state solutions of the system by utilizing the piecewise linearity of the governing equations which prevail before and after impact occurs. The authors, together with a colleague, D. L. Sikarskie, found that two such synchronous steady solutions were possible, but because of nonlinearity of the system (which gives rise to the nonunique solution) there is no assurance that either of the two predicted states can persist. A necessary condition for a given steady-state process to persist in a real system is that the system, in the presence of small random disturbances, shall stay sufficiently close to the given steady-state motion for all time. If, furthermore, the small disturbances, due to external forces, eventually fade away, the given steady-state motion is said to be asymptotically stable.

In this paper the asymptotic stability of the two-body system is considered. The analysis is parallel to a previous discussion of a similar problem, involving a one-degree-of-freedom system [1].

Regions in which stable steady-state solutions exist are determined. The stability regions provide guidance for a designer to choose design parameters which will result in a stable operating impact tool.

A separate computer solution has also been constructed to confirm the stability analysis.

2. EQUATION OF MOTION AND GENERAL SOLUTION

The equations of motion of the two-body system, between impacts, are,

$$m_1 \frac{d^2x_1}{dt^2} + K(x_1 - x_2) = P_0 + P_1 \cos(\omega t + \alpha) - H \tag{1}$$

$$m_2 \frac{d^2x_2}{dt^2} + K(x_2 - x_1) = -P_0 - P_1 \cos(\omega t + \alpha) - (m_1 + m_2)g \tag{2}$$

where all the quantities are defined in Fig. 1, except α which is an arbitrary constant to be defined later.

In terms of the nondimensional quantities

$$y_1 = Kx_1/P_1, \quad y_2 = Kx_2/P_1, \quad R = P_0/P_1, \quad F = H/P_1, \quad G_1 = m_1 g/P_1,$$

$$G_2 = m_2 g/P_1, \quad \mu = m_1/m_2, \quad \omega_0 = \left[\frac{K(m_1+m_2)}{m_1 m_2}\right]^{\frac{1}{2}}, \quad \lambda = \omega/\omega_0 \quad \text{and} \quad \tau = \omega_0 t,$$

the general solution to the above equations, for $\lambda \neq 1$, is

$$\begin{aligned} y_1 = {} & C_3 \sin\tau + C_4 \cos\tau + \frac{1}{(1+\mu)(1-\lambda^2)}\cos(\lambda\tau+\alpha) \\ & - \frac{\mu}{2(1+\mu)^2}(F+G_1+G_2)\tau^2 + C_1\tau + C_2 + C_0, \end{aligned} \tag{3}$$

$$\begin{aligned} y_2 = {} & -\mu\left[C_3 \sin\tau + C_4 \cos\tau + \frac{1}{(1+\mu)(1-\lambda^2)}\cos(\lambda\tau+\alpha) + C_0\right] \\ & - \frac{\mu}{2(1+\mu)^2}(F+G_1+G_2)\tau^2 + C_1\tau + C_2 \end{aligned} \tag{4}$$

where C_1, C_2, C_3, and C_4 are the four integration constants and,

$$C_0 = \frac{R+G_1}{1+\mu} - \frac{F}{(1+\mu)^2}.$$

3. SIMPLE STEADY-STATE SOLUTION

It will be assumed that the case never impacts against the fixed surface, and that a steady state motion is ultimately achieved. We are particularly interested in those steady-state motions which have the same period $(2\pi/\lambda)$ as the exciting force, and which exhibit only one impact per cycle of the exciting force. Such a "simple" steady-state solution can be constructed by further assuming that the impact occurs instantaneously and can be fully described by a single parameter e, which is referred to as the "effective coefficient of restitution" for the system. Therefore, it is assumed that upon impact the second mass is instantaneously stopped and has its speed changed from V to eV and its direction reversed. Conditions to be satisfied are:

$$\begin{aligned} y_2(0) &= 0 \\ \dot{y}_2(0) &= eV_{20} \qquad \left(\cdot \equiv \frac{d}{d\tau}\right) \\ y_1(0) &= y_{10} \\ \dot{y}_1(0) &= V_{10} \end{aligned} \tag{5}$$

$$\begin{aligned} y_2(2\pi/\lambda) &= 0 \\ \dot{y}_2(2\pi/\lambda) &= -V_{20} \\ y_1(2\pi/\lambda) &= y_{10} \\ \dot{y}_1(2\pi/\lambda) &= V_{10} \end{aligned} \tag{6}$$

where

$$V_{20} = VK/P_1\omega_0.$$

It is now clear that the value of α is determined by the condition that impact is assumed to occur at $\tau = 0$ and $2\pi/\lambda$.

A substituion of equations (3) and (4) into equations (5) gives,

$$\begin{aligned} C_1 &= \frac{1}{1+\mu}(eV_{20}+\mu V_{10}) \\ C_2 &= \frac{\mu}{1+\mu} y_{10} \\ C_3 &= \frac{1}{1+\mu}\left(V_{10} - eV_{20} + \frac{\lambda}{1-\lambda^2}\sin\alpha\right) \\ C_4 &= \frac{1}{1+\mu}\left(y_{10} - \frac{1}{1-\lambda^2}\cos\alpha\right) - C_0. \end{aligned} \tag{7}$$

Using equations (6) we can eliminate the unknown quantities y_{10}, V_{10}, and V_{20}, to obtain

$$\begin{aligned} C_1 &= \frac{\pi\mu}{\lambda(1+\mu)^2}(F+G_1+G_2) \\ C_2 &= \mu\left[-C_1 \cot\frac{\pi}{\lambda}+\frac{\cos\alpha}{(1+\mu)(1-\lambda^2)}+C_0\right] \\ C_3 &= -C_1 \\ C_4 &= -C_1 \cot\frac{\pi}{\lambda} \end{aligned} \tag{8}$$

$$\sin\alpha = -\frac{\pi(1-e)(1-\lambda^2)}{\lambda^2(1+e)}(F+G_1+G_2). \tag{9}$$

For the steady-state solution to exist, it is necessary that

$$\begin{aligned} &V_{20} \geq 0, \\ &y_1(\tau) \geq 0 \quad \text{for all } \tau, \\ &y_2(\tau) \geq 0 \quad \text{for all } \tau. \end{aligned} \tag{10}$$

Note that if $\lambda = 1/n$, where n is a positive integer greater than unity, we have $\cot(\pi/\lambda) \to \infty$ and the values of C_2 and C_4 approach infinity. However, solutions which correspond to $\lambda = 1/n$ are all inadmissible because of equation (10). Therefore resonance occurs only at $\lambda = 1$.

4. STABILITY OF SIMPLE STEADY-STATE SOLUTION

A given steady-state solution is said to be asymptotically stable or unstable if slight perturbations decay or grow as time goes on (i.e. time $\to \infty$).

Following [1], let us consider that the system has been perturbed by a small impulsive force which makes it depart momentarily from the simple steady-state solution. The perturbed solution will now be found.

Equation (7) shows the dependence of the constants C_i ($i = 1, 2, 3,$ and 4) on the initial values of α, V_{20}, y_{10}, and V_{10}. Let us consider small perturbations $\Delta\alpha$, ΔV_2, Δy_1, and ΔV_1 on the initial values. The motion of the perturbed state is given by equations (3) and (4), with C_i' replacing C_i, where

$$\begin{aligned} C_i' = C_i &+ \frac{\partial C_i}{\partial V_{10}}\Delta V_1 + \frac{\partial C_i}{\partial y_{10}}\Delta y_1 + \frac{\partial C_i}{\partial V_{20}}\Delta V_2 + \frac{\partial C_i}{\partial \alpha}\Delta\alpha \\ &+ \text{higher order terms} \end{aligned} \tag{11}$$

Using equation (7) and neglecting the higher order terms, we find

$$
\begin{aligned}
C_1' &= C_1+\frac{1}{1+\mu}(e\Delta V_2+\mu\Delta V_1)\\
C_2' &= C_2+\frac{\mu}{1+\mu}\Delta y_1\\
C_3' &= C_3+\frac{1}{1+\mu}\left(\Delta V_1-e\Delta V_2+\frac{\lambda}{1-\lambda^2}\cos\alpha\Delta\alpha\right)\\
C_4' &= C_4+\frac{1}{1+\mu}\left(\Delta y_1+\frac{1}{1-\lambda^2}\sin\alpha\Delta\alpha\right).
\end{aligned}
\tag{12}
$$

Equations (3) and (4), with C_i' replacing C_i, hold true until the time when the second body, m_2, strikes the fixed surface. Let this particular time be given by $\omega_0 t = \tau = T$; i.e.,

$$y_2(T) = 0, \tag{13}$$

$$y_1(T) = y_{10}+\Delta y_1' \tag{14}$$

$$\dot{y}_1(T) = V_{10}+\Delta V_1' \tag{15}$$

$$\dot{y}_2(T) = -(V_{20}+\Delta V_2'). \tag{16}$$

The value of α also changes to $\alpha+\Delta\alpha'$. Then,

$$T = \frac{2\pi}{\lambda}+\Delta\alpha'-\Delta\alpha. \tag{17}$$

Equations (13) to (16) give four equations for the four unknowns $\Delta\alpha'$, $\Delta V_2'$, $\Delta y_1'$, and $\Delta V_1'$ in terms of the original perturbations $\Delta\alpha$, ΔV_2, Δy_1, and ΔV_1. These equations, in general, contain trigonometric terms of arguments $\alpha+\Delta\alpha$, $\alpha+\Delta\alpha'$, $2\pi/\lambda+\Delta\alpha$, $2\pi/\lambda+\Delta\alpha'$, etc. If we expand all these terms in powers of $\Delta\alpha$ and $\Delta\alpha'$, we obtain the system of linear equations

$$
\begin{bmatrix}\Delta V_1'\\ \Delta y_1'\\ \Delta V_2'\\ \Delta\alpha'\end{bmatrix}
=
\begin{bmatrix}P_{11}P_{12}P_{13}P_{14}\\ P_{21}P_{22}P_{23}P_{24}\\ P_{31}P_{32}P_{33}P_{34}\\ P_{41}P_{42}P_{43}P_{44}\end{bmatrix}
\begin{bmatrix}\Delta V_1\\ \Delta y_1\\ \Delta V_2\\ \Delta\alpha\end{bmatrix}
\tag{18}
$$

Let us define:

$$
\begin{aligned}
B_1 &= \frac{1}{1+\mu}\sin\frac{2\pi}{\lambda}\\
B_2 &= \frac{1}{1+\mu}\left(\frac{2\pi}{\lambda}-\sin\frac{2\pi}{\lambda}\right)\\
B_3 &= \frac{1}{1+\mu}\left(\frac{2\pi}{\lambda\mu}+\sin\frac{2\pi}{\lambda}\right)\\
B_4 &= \frac{1}{1+\mu}\left(\frac{2\pi\mu}{\lambda}+\sin\frac{2\pi}{\lambda}\right)
\end{aligned}
$$

$$B_5 = \frac{1}{1+\mu}\left(1-\cos\frac{2\pi}{\lambda}\right)$$

$$B_6 = \frac{1}{1+\mu}\left(\mu+\cos\frac{2\pi}{\lambda}\right)$$

$$B_7 = \frac{1}{1+\mu}\left(\frac{1}{\mu}+\cos\frac{2\pi}{\lambda}\right)$$

$$B_8 = \frac{1}{1-\lambda^2}(\lambda B_1 \cos\alpha - B_5 \sin\alpha)$$

$$B_9 = \frac{1}{1-\lambda^2}(B_1 \sin\alpha + \lambda B_5 \cos\alpha)$$

$$P_{10} = \frac{\lambda\cos\alpha}{(1+\mu)(1-\lambda^2)} + C_1\left(\frac{1}{\pi}-\frac{1}{\lambda}\cot\frac{\pi}{\lambda}\right)$$

$$P_{20} = \frac{(1+\mu)(1-e)}{\lambda\mu(1+e)}C_1$$

$$P_{30} = \frac{\lambda\mu\cos\alpha}{(1+\mu)(1-\lambda^2)} - C_1\left(\frac{1}{\pi}+\frac{\mu}{\lambda}\cot\frac{\pi}{\lambda}\right)$$

$$P_{40} = \frac{\lambda(1+e)}{2(1+\mu)C_1}$$

We find,

$$\begin{aligned}
P_{41} &= \lambda\mu B_2 P_{40} \\
P_{42} &= \lambda\mu B_5 P_{40} \\
P_{43} &= \lambda e\mu B_3 P_{40} \\
P_{44} &= 1-\mu B_8 P_{40} \\
P_{31} &= -\mu B_5 - P_{30}P_{41} \\
P_{32} &= \mu B_1 - P_{30}P_{42} \\
P_{33} &= e\mu B_7 - P_{30}P_{43} \\
P_{34} &= \frac{1}{\lambda}[-\mu B_9 + P_{30}(1-P_{44})] \\
P_{21} &= B_4 + P_{20}P_{41} \\
P_{22} &= B_6 + P_{20}P_{42} \\
P_{23} &= eB_2 + P_{20}P_{43} \\
P_{24} &= \frac{1}{\lambda}[B_8 + P_{20}(P_{44}-1)] \\
P_{11} &= B_6 - P_{10}P_{41}
\end{aligned} \qquad (19)$$

$$P_{12} = -B_1 - P_{10}P_{42}$$

$$P_{13} = eB_5 - P_{10}P_{43}$$

$$P_{14} = \frac{1}{\lambda}[-B_9 + P_{10}(1 - P_{44})].$$

We now think of the initial perturbations ΔV_1, Δy_1, ΔV_2, and $\Delta\alpha$ as components of a vector. After the first impact the perturbations for the next cycle of motion are given by a vector of components $(\Delta V_1', \Delta y_1', \Delta V_2', \Delta\alpha')$, which are determined from equations (18), i.e.,

$$\{\Delta V_1', \Delta y_1', \Delta V_2', \Delta\alpha'\} = [P]\{\Delta V_1, \Delta y_1, \Delta V_2, \Delta\alpha\};$$

after the second impact they are given by,

$$\{\Delta V_1'', \Delta y_1'', \Delta V_2'', \Delta\alpha''\} = [P]\{\Delta V_1', \Delta y_1', \Delta V_2', \Delta\alpha'\}$$
$$= [P]^2\{\Delta V_1, \Delta y_1, \Delta V_2, \Delta\alpha\},$$

etc. If the vector $[P]^n\{\Delta V_1, \Delta y_1, \Delta V_2, \Delta\alpha\}$ tends to zero as n approaches infinity, the perturbed solution approaches the steady-state solution and the solution is said to be asymptotically stable. It should be observed that only first order perturbation terms were retained in equation (11). However, Masri and Caughey [2] proved that the first order terms do indeed govern the asymptotic stability of systems of this type. Using a proof parallel to that of [1] and [2], one can show that asymptotic stability is insured by the condition that the modulus of each eigenvalue, Q_i, of the matrix P is less than one and instability follows from the condition that at least one of the moduli of the eigenvalues is greater than one, i.e., the solution is asymptotically stable, if

$$|Q_i| < 1 \quad \text{for } i = 1, 2, 3, \text{and } 4$$

and asymptotically unstable, if

$$|Q_i| > 1 \quad \text{for } i = 1, 2, 3, \text{or } 4.$$

Q_i are determined by the equation

$$|P_{ij} - Q\delta_{ij}| = 0. \tag{20}$$

5. NUMERICAL EXAMPLES AND DISCUSSION

As a numerical example we choose the parameters: $F = 0{\cdot}45$, $R = 1{\cdot}16$, $G_1 = 0{\cdot}331$, and $G_2 = 0{\cdot}029$, and seek regions of the e–λ plane in which the analytically constructed solutions are asymptotically stable.

It has been pointed out that for a given set of values of e and λ, there exists, in general, two branches of solutions. This is simply because the value of α enters into the analysis through equation (9) in the form of $\sin\alpha$, which does not define the value of α uniquely. If α_0 is a solution, then $\pi - \alpha_0$ is also a solution. Let us assume that the first branch of solutions is associated with α_0, where $-\pi/2 \leq \alpha_0 \leq \pi/2$, and the second branch with $\pi - \alpha_0$.

By examining the eigenvalues we have found that the first branch of the two solutions is always unstable, for all values of e and λ investigated, $(0 \leq \lambda \leq 5;\ 0 \leq e \leq 1)$. For the second branch of the solution, narrow stability regions in the e–λ plane have been found and are shown in Fig. 2. These numerical results were carried out by a digital computer.

The eigenvalues of the matrix P are determined using a subroutine obtained from SHARE [3].

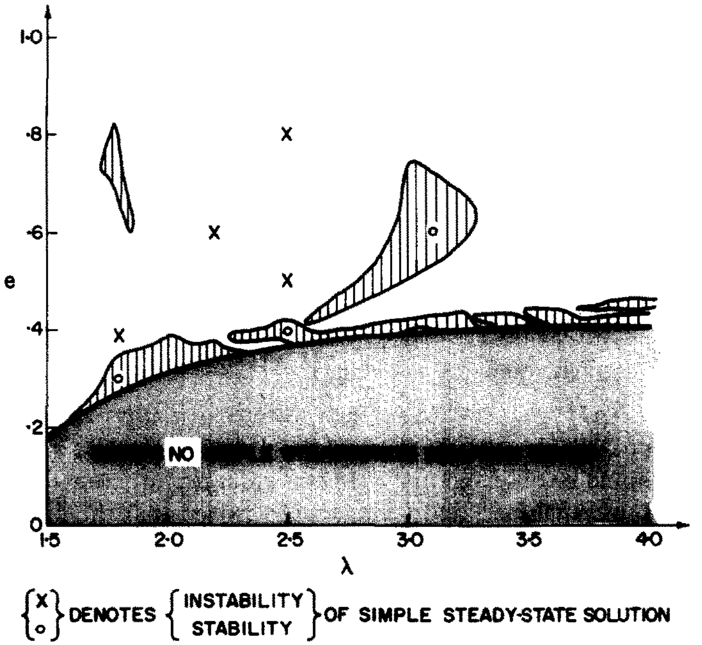

FIG. 2. Regions of stable simple steady-state solutions.

(Stable regions shown shaded with vertical lines are found from stability analysis. Points x, o are found from the computer solution of the initial value problem.)

6. A COMPUTER SOLUTION

A computer program was written to directly determine the four integration constants of equations (3) and (4). First, one starts with an arbitrary set of initial values which determines the four constants. The motion is then followed until the time when $y_2 = 0$ occurs. The values of $\dot{y}_1$, y_1, and $\dot{y}_2$ are next computed at that particular time. Multiplying the value of $\dot{y}_2$ by $(-e)$ together with the other calculated values provide a set of new initial conditions from which one can determine the four constants C_i of equations (3) and (4) anew. The process is repeated over and over so as to obtain the time behavior of the system.

The computer solution can be used to provide a check on the stability results. Several points, as shown in Fig. 2, have been checked and the computer results agree with the theoretical predictions on stability.

Here we point out, however, that the stability of the system is very "weak" in the sense that a solution is stable only for extremely small perturbations.

7. CONCLUSIONS

Asymptotic stability of the simple steady-state motion for a two-body dynamic system subjected to repeated impact conditions has been formulated and analyzed. Narrow stability regions have been found. A separate computer solution, however, indicated that all the stable solutions stay stable only for extremely small perturbations. Therefore, we conclude that, although narrow stability regions can be found, the simple steady-state motion of the system is essentially unstable from a practical point of view. If one wishes to design such a "floating" system, one should not expect a steady-state motion which has the same period as the exciting force and which exhibits only one impact per cycle of the steady-state motion.

Practical impact tools incorporate "stops" on the case which will impact the bit (or "fixed surface", in our idealization) from time to time, and may have a wider range of stable operating conditions than the "floating" system which was analyzed in this paper.

REFERENCES

[1] C. C. Fu and B. Paul, Dynamic stability of a vibrating hammer. To be published.
[2] S. F. Masri and T. K. Caughey, On the stability of the impact damper. *J. appl. Mech.* 586–592 (Sept. 1966).
[3] B. Parlett, Laguerrés method applied to the matrix eigenvalue problem. *Maths. Comput.* **18**, 464–485 (1964).

(*Received* 26 *December* 1967; *revised* 26 *February* 1968)

Абстракт—Определяется метод расчета динамики твердого тела для машин, в которых молот имеет возвратно-поступательное движение внутри корпуса и делает периодически удар на сверло, долотчитый бур, наковальню, или на другой член, поглащающий энергию. Идеалиризуется физическая система, которая представляет собой "плавающую" модель двух тел. В данном случае могут действовать два простые усмановивмиеся движения. Определяется, также, что плавающая модель двух тел допускает теоретически одно устойчивое усмановивмееся решение, которое обладает таким же периодом колебаний как и вынуждающая сила и допускает только один удар в цикл усмановивмегося движения. Определяются районы устойчивости простых усмановивмихся решений.

Найдено отдельное решение для вычислительных машни, с целью определения детальной истории движения системы. Результаты указывают, что аналитические усменовивмиеся решения являются устойчивыми только для крайне малых возмущений.